Scale-Locality of Magnetohydrodynamic Turbulence

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We investigate the scale-locality of cascades of conserved invariants at high kinetic and magnetic Reynolds numbers in the "inertial-inductive range" of magnetohydrodynamic (MHD) turbulence, where velocity and magnetic field increments exhibit suitable power-law scaling. We prove that fluxes of total energy and cross-helicity—or, equivalently, fluxes of Elsässer energies—are dominated by the contributions of local triads. Corresponding spectral transfers are also scale-local when defined using octave wavenumber bands. Flux and transfer of magnetic helicity may be dominated by non-local triads. The magnetic stretching term also may be dominated by non-local triads but we prove that it can convert energy only between velocity and magnetic modes at comparable scales. We explain the disagreement with numerical studies that have claimed conversion nonlocally between disparate scales. We present supporting data from a 1024^3 simulation of forced MHD turbulence.

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Magnetohydrodynamic (MHD) turbulence is pervasive in astrophysical systems. Turbulent plasma fluctuations commonly possess power-law spectra over vast ranges of scales where both viscosity and resistivity are negligible. We call such ranges "inertial-inductive", since nonlinear dynamics (inertia/Lorentz force and convection/induction) dominates the physics at these scales. MHD plasma turbulence, with power-law scaling of both spectra and structure-functions in the inertial-inductive range, plays a central role in star formation, accretion of matter near active galactic nuclei, solar physics, and in the generation of large-scale magnetic fields in such systems. There are several competing theories for the spectrum of strong MHD turbulence, including those of Iroshnikov-Kraichnan [1, 2], Goldreich-Sridhar [3], and Boldyrev [4]. All of these theories assume scale-locality of the nonlinear cascade, following the classical ideas of Richardson, Kolmogorov and Onsager for turbulence in neutral fluids. Scale-locality is fundamental to justify the universality of the postulated turbulent scaling laws.

A consensus has been forming in recent years, however, that cascades in MHD turbulence are nonlocal processes [5, 6, 7, 8]. Schekochihin et al. [5] emphasized the nonlocal nature of the interactions between the velocity and magnetic fields as a hallmark of isotropic MHD turbulence. This conclusion was reaffirmed in several subsequent studies, most categorically by Yousef et al. [8] who claimed that there is a direct exchange of energy between motions at the largest scales in the system, at which the flow is being stirred, and the magnetic field at arbitrarily small scales in the inductive range. A more refined analysis of the locality of interactions was carried out by Alexakis et al. [6] who concluded, based on direct numerical simulations (DNS) of MHD turbulence that the magnetic field gains energy at scales ℓ in the inertial range from the straining motions at all larger scales $> \ell$ and especially from the forcing scale $L \gg \ell$. Carati et al. [7] subsequently carried out DNS at a higher resolution, arriving at conclusions similar to [6] and, furthermore, claiming that there is non-local transfer of Elsässer energies as well. Related ideas have surfaced in the accretion disk community [9, 10].

In this letter we address the scale-locality of MHD cascades by a direct analytical study of the equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b} + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\partial_t \mathbf{b} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{b} = (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{u} + \eta\nabla^2\mathbf{b} \qquad (1)$$

for $\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{b} = 0$. Here $\mathbf{b} = \mathbf{B}/\sqrt{4\pi\rho}$ is the magnetic field in Alfvén velocity units and p is total pressure (including magnetic pressure). Our main conclusion is that, under very weak scaling assumptions, MHD turbulence has scale-locality properties only a little less robust than those of hydrodynamic turbulence. We will support our analysis with a pseudospectral DNS at 1024^3 resolution with phase-shift dealiasing. For our numerical work we choose viscosity ν and resistivity η to be both equal to 1.1×10^{-4}. The external stirring force is a Taylor-Green flow $\mathbf{f} \equiv f_0[\sin(k_f x)\cos(k_f y)\cos(k_f z)\hat{\mathbf{x}} - \cos(k_f x)\sin(k_f y)\cos(k_f z)\hat{\mathbf{y}}]$ applied at modes $k_f = 2$ with an amplitude $f_0 = 0.25$. The Reynold's number based on the Taylor scale $\lambda_u = 2\pi\sqrt{E_u}/\big(\int dk k^2 E_u(k)\big)^{1/2}$ is $Re_{\lambda_u} = u_{rms}\lambda_u/\nu = 909$.

Our proof of local cascade of invariants in MHD turbulence is very similar to that given for hydrodynamic turbulence in [11, 12, 13]. We employ the spatial coarse-graining approach, commonly used as a modelling tool in the Large-Eddy Simulation (LES) community [14, 15]. Coarse-grained fields are defined by $\overline{f}_\ell(\mathbf{x}) = \int d\mathbf{r}\, G_\ell(\mathbf{r}) f(\mathbf{x}+\mathbf{r})$, with a filtering kernel $G_\ell(\mathbf{r}) = \ell^{-3}G(\mathbf{r}/\ell)$ which is sufficiently smooth and decays sufficiently rapidly for large r [11]. Coarse-grained MHD equations can then be written to describe $\overline{\mathbf{u}}_\ell$ and $\overline{\mathbf{b}}_\ell$, along with corresponding budgets for the quadratic invariants—energy, cross-helicity, and

magnetic-helicity—at scales $\geq \ell$. See [16]. E.g. the time-derivative of large-scale energy $(1/2)[|\overline{\mathbf{u}}_\ell|^2 + |\overline{\mathbf{b}}_\ell|^2]$, in addition to space-transport terms, contains also as sink terms the *kinetic energy flux* $-\Pi^u_\ell = \boldsymbol{\nabla}\overline{\mathbf{u}}_\ell : \boldsymbol{\tau}_\ell$ and the *magnetic energy flux* $-\Pi^b_\ell = \overline{\mathbf{j}}_\ell \boldsymbol{\cdot} \boldsymbol{\varepsilon}_\ell$, with $\overline{\mathbf{j}}_\ell = \boldsymbol{\nabla}\times\overline{\mathbf{b}}_\ell$. Here $\tau_{\ell,\,ij} = \tau_\ell(u_i, u_j) - \tau_\ell(b_i, b_j)$ is the total stress generated by scales $< \ell$, both the Reynolds stress and the Maxwell stress, and $\varepsilon_{\ell,\,i} = \epsilon_{ijk}\tau_\ell(u_j, b_k)$ is the electromotive force generated by scales $< \ell$. We employ the notation

$$\tau_\ell(f, g) = \overline{(fg)_\ell} - \overline{f}_\ell \overline{g}_\ell \tag{2}$$

for the "central moments" of any fields $f(\mathbf{x}), g(\mathbf{x})$ [14].

There are two facts crucial for scale-locality of the energy fluxes $\Pi^{u,b}_\ell$. First, all the filtered gradient-fields and the central moments can be expressed in terms of *increments*. In general, for any fields,

$$\boldsymbol{\nabla}\overline{f}_\ell \approx \delta f(\ell)/\ell, \;\; \tau_\ell(f, g) \approx \delta f(\ell)\delta g(\ell), \;\; f'_\ell \approx -\delta f(\ell) \tag{3}$$

where increments are $\delta f(\mathbf{x}, \mathbf{r}) = f(\mathbf{x}+\mathbf{r}) - f(\mathbf{x})$, $\delta f(\ell) = \sup_{r<\ell} |\delta f(\mathbf{r})|$, and $f'_\ell = f - \overline{f}_\ell$ is the fine-scale (high-pass filtered) field. For details, see [11]. The second crucial ingredient for locality is the scaling properties of the increments of velocity and magnetic field:

$$\delta u(\ell) \simeq \ell^{\sigma_u}, \quad \delta b(\ell) \simeq \ell^{\sigma_b}, \quad 0 < \sigma_{u,b} < 1, \tag{4}$$

where these relations may be assumed to hold either pointwise, with σ the local Hölder exponent, or in the sense of pth-order means, $\|\delta f\|_p = \langle |\delta f(\ell)|^p \rangle^{1/p}$, with σ equal to $1/p$ times the scaling-exponent ζ_p of the pth-order structure function. As long as $0 < \sigma_{u,b} < 1$, then (either locally or in the L_p-mean sense) the fluxes $\Pi^{u,b}_\ell$ are determined by modes all at scales comparable to ℓ [11]. For example, the contribution to any increment $\delta f(\ell)$ from scales $\Delta \geq \ell$ is represented by $\delta\overline{f}_\Delta(\ell)$. Since the low-pass filtered field $\overline{f}_\Delta$ is smooth, its increment may be estimated by Taylor expansion and (3),(4) as

$$\delta\overline{f}_\Delta(\ell) \simeq \boldsymbol{\ell}\boldsymbol{\cdot}(\boldsymbol{\nabla}\overline{f}_\Delta) \simeq \ell\Delta^{\sigma-1} \simeq \ell^\sigma(\ell/\Delta)^{1-\sigma},$$

and this is negligible for $\Delta \gg \ell$ as long as $\sigma < 1$. On the other hand, the contribution to any increment $\delta f(\ell)$ from scales $\delta \leq \ell$ is represented by $\delta f'_\delta(\ell)$. Since $f'_\delta \approx -(\delta f)(\delta)$ (even without taking any difference), (4) implies that

$$\delta f'_\delta(\ell) \simeq \delta^\sigma \simeq \ell^\sigma(\delta/\ell)^\sigma,$$

and this is negligible for $\delta \ll \ell$ as long as $\sigma > 0$.

It is important to emphasize that the scaling laws like (4) used in our proof are obtained in all theories of strong MHD turbulence. The Iroshnikov-Kraichnan theory predicts that $\sigma_u = \sigma_b = 1/4$. The Goldreich-Sridhar theory predicts distinct scaling for increments with displacements in different directions relative to a background field $\mathbf{b}_0$, with $\delta u(\ell_\parallel) \sim \delta b(\ell_\parallel) \sim \ell_\parallel^{1/2}$ for displacements in the field-parallel direction and $\delta u(\ell_\perp) \sim \delta b(\ell_\perp) \sim \ell_\perp^{1/3}$ for displacements in the perpendicular direction. Such distinctions make no difference to our proof, so long as both exponents $\sigma_\parallel$, $\sigma_\perp$ lie between 0 and 1. Similarly, our proof is fully compatible with possible intermittency corrections to scaling exponents. Although the precise scaling of strong MHD turbulence is an open issue, numerical simulations [17, 18] and natural observations [19, 20] support the validity of the weak condition (4) for sufficiently high kinetic and magnetic Reynolds numbers.

Our arguments imply also the scale-locality of cascades of the Elsässer energies $(1/2)|\mathbf{z}^\pm|^2$, with $\mathbf{z}^\pm = \mathbf{u} \mp \mathbf{b}$. This may be seen by considering the time-derivative of the large-scale energy densities $(1/2)|\overline{\mathbf{z}}^\pm_\ell|^2$, for which the sink terms are the fluxes $-\Pi^\pm_\ell = \boldsymbol{\nabla}\overline{\mathbf{z}}^\pm_\ell \boldsymbol{:} \tau_\ell(\mathbf{z}^\mp, \mathbf{z}^\pm) \simeq \delta z^\mp(\ell)[\delta z^\pm(\ell)]^2/\ell$. Since these fluxes are expressed in terms of increments, they are scale-local under the weak condition (4). This may also be seen from an alternative expression for the Elsässer energy fluxes which follow from the Politano-Pouquet relations [21], $\Pi^\pm_\ell = -(3/4\ell)\langle \hat{\boldsymbol{\ell}}\boldsymbol{\cdot}\delta\mathbf{z}^\mp(\boldsymbol{\ell})|\delta\mathbf{z}^\pm(\boldsymbol{\ell})|^2\rangle_{\rm ang}$, where $\langle\cdot\rangle_{\rm ang}$ denotes average over the displacement directions $\hat{\boldsymbol{\ell}}$. The scale-locality of cascades of the Elsässer energies is particularly important since the foremost phenomenologies of strong MHD turbulence [1, 2, 3, 4] are based on the picture of counterpropagating Alfvén wavepackets expressed by the Elsässer variables $\mathbf{z}^\pm$. In terms of these variables, the scale-locality properties of MHD turbulence are essentially the same as those of hydrodynamic turbulence. Scale-locality of the cascades of Elsässer energies implies scale-locality of the flux of cross-helicity $\overline{\mathbf{u}}_\ell \boldsymbol{\cdot} \overline{\mathbf{b}}_\ell = (1/4)|\overline{\mathbf{z}}^+_\ell|^2 - (1/4)|\overline{\mathbf{z}}^-_\ell|^2$ (as well as scale-locality of flux of total energy $(1/4)|\overline{\mathbf{z}}^+_\ell|^2 + (1/4)|\overline{\mathbf{z}}^-_\ell|^2$).

One cascade in MHD turbulence which may be essentially different is that of magnetic helicity. The time-derivative of large-scale helicity-density $\overline{\mathbf{b}}_\ell \boldsymbol{\cdot} \overline{\mathbf{a}}_\ell$ (where $\overline{\mathbf{a}}_\ell = (\mathrm{curl})^{-1}\overline{\mathbf{b}}_\ell$), in addition to space-transport terms, contains as a sink term the *magnetic helicity flux* $-\Pi^h_\ell = 2\overline{\mathbf{b}}_\ell \boldsymbol{\cdot} \boldsymbol{\varepsilon}_\ell$. Although $\boldsymbol{\varepsilon}_\ell \simeq \delta u(\ell)\delta b(\ell)$, the coarse-grained magnetic field $\overline{\mathbf{b}}_\ell$ will generally be dominated by modes at the forcing scale L. Thus, magnetic-helicity flux may possibly be dominated by non-local triads, with one mode at the large scale L. Similar issues arise for magnetic line-stretching. The time-derivative of large-scale kinetic energy $(1/2)|\overline{\mathbf{u}}_\ell|^2$, in addition to space-transport terms and the sink term $-\Pi^u_\ell$, contains $-\overline{\mathbf{b}}_\ell^\top \overline{\mathbf{S}}_\ell \overline{\mathbf{b}}_\ell$ where the matrix $\overline{\mathbf{S}}_\ell = (1/2)[(\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell) + (\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell)^\top]$ is the strain from scales $> \ell$. Likewise, the time-derivative of large-scale magnetic energy $(1/2)|\overline{\mathbf{b}}_\ell|^2$, in addition to space-transport terms and the sink term $-\Pi^b_\ell$, contains $+\overline{\mathbf{b}}_\ell^\top \overline{\mathbf{S}}_\ell \overline{\mathbf{b}}_\ell$. Thus, this term represents conversion between large-scale kinetic and magnetic energy by stretching of coarse-grained field-lines. Just as for magnetic helicity flux, this is a "hybrid" quantity with both energy-range and inertial-range components. Although $\overline{\mathbf{S}}_\ell \sim \delta u(\ell)/\ell$, the coarse-grained

magnetic field $\overline{\mathbf{b}}_\ell$ can be dominated by modes at scale L. Thus, we cannot conclude that this quantity is dominated by local triads with all modes at scale ℓ.

Indeed, much of the recent discussion about apparent non-locality in MHD turbulence has revolved about this conversion term. One of the startling claims that has been made in recent numerical studies [6, 7, 8] is that conversion between kinetic and magnetic energies proceeds very non-locally, with magnetic modes at scale ℓ gaining energy equally from all velocity modes at scales $> \ell$ or even predominately from scale $L \gg \ell$. In order to examine this claim, we must refine our methodology to consider *band-pass energies*. Following [12], we define pointwise the kinetic and magnetic energy densities in the interval of scales $[\overline{\ell}, \widetilde{\ell}]$ for $\widetilde{\ell} > \overline{\ell}$, as

$$e^u_{[\overline{\ell},\widetilde{\ell}]} = (1/2)\widetilde{\tau}(\overline{u}_i, \overline{u}_i), \quad e^b_{[\overline{\ell},\widetilde{\ell}]} = (1/2)\widetilde{\tau}(\overline{b}_i, \overline{b}_i).$$

Note that $\overline{(\cdot)}$ now denotes scale $\overline{\ell}$ and $\widetilde{(\cdot)}$ scale $\widetilde{\ell}$. Their time-derivatives are easily calculated to be

$$\partial_t e^u_{[\overline{\ell},\widetilde{\ell}]} = -(\widetilde{\overline{b}_i\overline{b}_j\overline{S}_{ij}} - \widetilde{\overline{b}}_i\widetilde{\overline{b}}_j\widetilde{\overline{S}}_{ij}) + \left(\widetilde{\overline{\Pi}}^u - \widetilde{(\overline{\Pi}^u)}\right) + \cdots \quad (5)$$

$$\partial_t e^b_{[\overline{\ell},\widetilde{\ell}]} = +(\widetilde{\overline{b}_i\overline{b}_j\overline{S}_{ij}} - \widetilde{\overline{b}}_i\widetilde{\overline{b}}_j\widetilde{\overline{S}}_{ij}) + \left(\widetilde{\overline{\Pi}}^b - \widetilde{(\overline{\Pi}^b)}\right) + \cdots \quad (6)$$

where $\cdots$ denotes total divergence terms that correspond to space-transport. As before, $\overline{\Pi}^u = -\overline{\mathbf{S}}{:}\boldsymbol{\overline{\tau}}$ and $\overline{\Pi}^b = -\overline{\mathbf{j}}{\cdot}\overline{\boldsymbol{\varepsilon}}$, and note that the double-filtering length-scale $\widetilde{\overline{\ell}} \approx \widetilde{\ell}$ for $\widetilde{\ell} \gg \overline{\ell}$. It is "obvious" from these equations that the magnetic stretching terms transfer energy between velocity and magnetic-field modes only within the same band of length-scales $[\overline{\ell}, \widetilde{\ell}]$. Clearly, whatever energy is lost or gained from one field by line-stretching reappears in or disappears from the other field at the same scale. Non-colliding Alfvén waves are an example of such non-local triadic exchange which is mediated by a uniform magnetic field at the largest scales, but which does not contribute to energy transfer across scales.

Our conclusion above requires some caution, however. A counterexample is the Batchelor (viscous-inductive) range that occurs in MHD turbulence with a large magnetic Prandtl number $Pr_m = \nu/\eta \gg 1$ [22]. This range consists of length-scales $\ell_\nu \gg \ell \gg \ell_\eta$ far below the inertial-inductive range $L \gg \ell \gg \ell_\nu$, with ℓ_ν and ℓ_η the viscous and resistive length-scales, resp. In the Batchelor range, the energy is transferred directly from the velocity modes at the viscous scale ℓ_ν into the magnetic-field modes at scales $\ell \ll \ell_\nu$. To see that this follows from our eqs. (5)-(6), we observe that the velocity-gradient in the Batchelor range is almost spatially constant and $\boldsymbol{\nabla}\overline{\mathbf{u}}_\ell \approx \boldsymbol{\nabla}\mathbf{u}$ for all $\ell < \ell_\nu$. It is thus easy to see that the stretching term in (5) equals $-S_{ij}\widetilde{\tau}(\overline{b}_i, \overline{b}_j)$ whereas the two flux terms become $S_{ij}(\widetilde{\overline{\tau}(b_i, b_j)} - \widetilde{\overline{\tau}}(b_i, b_j))$. (Note that the stress in this range is almost entirely Maxwellian). These terms exactly cancel, by the Germano identity [14, 15]. Thus, the line-stretching term acts as an effective source to magnetic energy $e^b_{[\overline{\ell},\widetilde{\ell}]}$, supplied by the flux of kinetic energy directly from the viscous scale ℓ_ν.

The moral of this example is that the energy fluxes also contain line-stretching effects which must be considered. Nevertheless, our conclusion is not altered that, *in an inertial-inductive range*, energy conversion by line-stretching is between velocity and magnetic-field modes at similar scales. The key point here is the scale-locality of the fluxes, which has already been established. Because the fluxes only involve modes at comparable scales, they cannot transfer energy from very distant scales into scale ℓ within an inertial-inductive range. This is not true in the Batchelor range since the velocity field is very smooth there ($\sigma_u = 1$), violating the condition (4) for scale-locality of energy flux in the infrared.

The studies [6, 7, 8] considered more traditional spectral transfers such as $T_{ub}(K, P) = \langle \mathbf{b}^{[P]}(\mathbf{b}{\cdot}\boldsymbol{\nabla})\mathbf{u}^{[K]}\rangle$ and $T_{bu}(K, P) = \langle \mathbf{u}^{[P]}(\mathbf{b}{\cdot}\boldsymbol{\nabla})\mathbf{b}^{[K]}\rangle$, where $\mathbf{u}^{[K]}$ and $\mathbf{b}^{[K]}$ are spectrally band-passed fields for some interval of wavenumbers around K. Since $T_{ub}(K, P) = -T_{bu}(P, K)$, these can be interpreted (with some caution) as energy transfer from the velocity field in band $[K]$ to the magnetic field in band $[P]$. Is it possible for the dominant transfers to be between distant bands in an inertial-inductive range? The answer is no, if $[K]$ is a dyadic (octave) wavenumber band $[K/2, K]$. It is necessary to use such bands, of equal width on a logarithmic scale, in order to permit simultaneous localization of modes in Fourier and physical space (within the limits of the uncertainty principle). We note that this is crucial for phenomenological arguments based upon Alfvénic wavepackets with both size and wavenumber specified. The conditions which replace (4) are, for $\mathbf{a} = \mathbf{u}, \mathbf{b}$ with $0 < \sigma^a_p < 1$:

$$\langle |\mathbf{a}^{[K]}|^p\rangle^{1/p} \simeq K^{-\sigma^a_p}, \quad \langle \left|\boldsymbol{\nabla}\mathbf{a}^{[K]}\right|^p\rangle^{1/p} \simeq K^{1-\sigma^a_p}. \quad (7)$$

See [13]. If $P < K/2$, then wavenumber conservation implies that $T_{ub}(K, P) = -\langle \mathbf{u}^{[K]}(\mathbf{b}^{[K/2-P,K+P]}{\cdot}\boldsymbol{\nabla})\mathbf{b}^{[P]}\rangle$. Using the Hölder inequality, this expression is bounded by $\langle|\boldsymbol{\nabla}\mathbf{b}^{[P]}|^3\rangle^{1/3}\langle|\mathbf{u}^{[K]}|^3\rangle^{1/3}\langle|\mathbf{b}^{[K/2-P,K+P]}|^3\rangle^{1/3}$. By (7)

$$|T_{ub}(K, P)| \leq (\text{const.})P^{1-\sigma^b_3}K^{-\sigma^u_3-\sigma^b_3}$$

Since $\sigma^b_3 < 1$, such transfers for $P \ll K$ are negligible. For $P > 2K$, $T_{ub}(K, P) = \langle \mathbf{b}^{[P]}(\mathbf{b}^{[P/2-K,P+K]}{\cdot}\boldsymbol{\nabla})\mathbf{u}^{[K]}\rangle$, so Hölder inequality and (7) imply

$$|T_{ub}(K, P)| \leq (\text{const.})K^{1-\sigma^u_3}P^{-2\sigma^b_3}.$$

Since $\sigma^b_3 > 0$, transfers for $P \gg K$ are also negligible.

To test these conclusions numerically we analyze a time snapshot of our 1024^3 MHD simulation in the statistical steady state. The kinetic and magnetic energy spectra of the flow, have a reasonable power-law scaling until

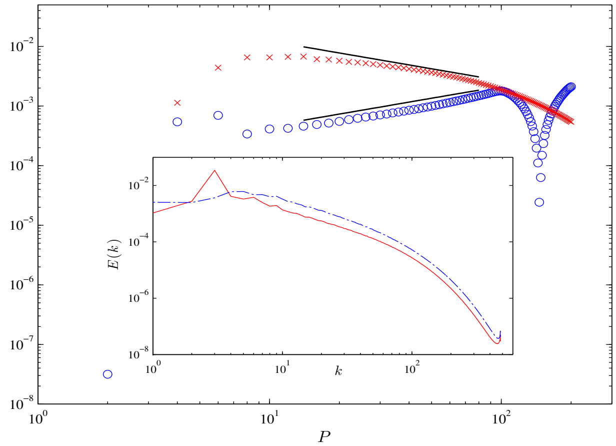

FIG. 1: The transfers $\left|\langle \partial_j u_i^{[P]} B_i^{[200]} B_j \rangle\right|$ (∘) and $\left|\langle \partial_j u_i^{[4]} B_i^{[P]} B_j \rangle\right|$ (×). Straight lines have ±2/3-slopes and extend over the fitting range, which yields a decay rate of $\sim P^{0.68}$ for (∘) and $\sim P^{-0.58}$ for (×). Inset shows velocity (solid line) and magnetic (dashed-dotted line) energy spectra, which scale close to $E_u \sim E_b \sim k^{-1.61}$ over $k \in [5, 80]$.

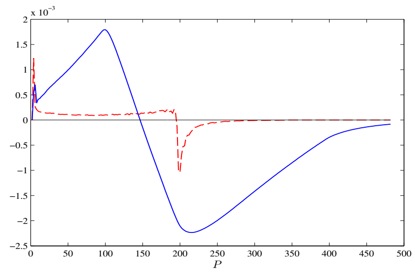

FIG. 2: The transfers $\langle \partial_j u_i^{[P/2,P]} B_i^{[100,200]} B_j \rangle$ (solid line, same as (∘) plot in Fig. 1) and $10^3 \times \langle \partial_j u_i^{[P-1,P]} B_i^{[199,200]} B_j \rangle$ (dashed line). The latter is multiplied by 1000 for comparison.

around $k = 80$ (inset to Fig. 1). The transfers plotted in Fig. 1 exhibit off-diagonal ($P \neq K$) decay close to our rigorous upper bounds with exponent $\sigma_3^b \doteq 1/3$. However, the value of this exponent determined from our numerical data (not shown) is closer to $\sigma_3^b = 1/4$, consistent with the predictions of [1, 2, 4]. For this value we obtain rigorous upper bounds $O(P^{0.75})$ for $P \ll K$ and $O(P^{-.5})$ for $P \gg K$, which are also close to the observed scaling.

How are our exact results to be reconciled with the recent numerical studies that reach the opposite conclusion? A full discussion is given in our longer work [16], but we make a few remarks here. [5, 8] discussed simulations at lower resolution than ours without carrying out a systematic scaling analysis. As for [7], they had an anomalously strong strain at the forcing scale L, which can dominate over the local strain at scales $\ell \lesssim L$ in an inertial-inductive range of limited extent. We also observe this effect over a finite range if we permit such an "energy spike" at the forcing scale, but it becomes weaker as the amplitude of the spike decreases or as the length of the power-law scaling range increases. Finally, [6] appealed to spectral transfers to justify their claim that the magnetic field at scales ℓ in the inertial-inductive range receives energy from straining motions at all larger scales $> \ell$, especially from scale $L \gg \ell$. However, their DNS study used Fourier bands of linear size $[K-1, K]$, which correspond to plane-wave modes which are nonlocalized in space, unlike the Alfvén wavepackets employed in phenomenological arguments. Such bands do not properly account for the exponentially growing number of local triads at higher wavenumbers, whose aggregate contribution dominates transfers defined with logarithmic bands [13]. Fig. 2 reproduces the numerical result of Fig. 8 of [6] (dashed-line), together with our own DNS results using log-bands. Clearly, the nonlocal effects observed by [6] represent miniscule amounts of energy transfer compared with the net contribution of local triads and become even smaller as the scale range increases. In short, the numerical results in [5, 6, 7, 8] do not support any asymptotic non-locality of energy cascade in MHD turbulence.

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[1] P. S. Iroshnikov, Sov. Astron. **7**, 566 (1964).
[2] R. H. Kraichnan, Phys. Fluids **8**, 1385 (1965).
[3] P. Goldreich and S. Sridhar, Astrophys. J. **438**, 763 (1995).
[4] S. Boldyrev, Astrophys. J. **626**, L37 (2005).
[5] A. A. Schekochihin et al., Astrophys. J. **612**, 276 (2004).
[6] A. Alexakis, P. D. Mininni, and A. Pouquet, Phys. Rev. E **72**, 046301 (2005).
[7] D. Carati et al., J. Turb. **7**, 1 (2006).
[8] T. A. Yousef, F. Rincon, and A. A. Schekochihin, J. Fluid Mech. **575**, 111 (2007).
[9] S. Fromang and J. Papaloizou, Astron. Astrophys. **476**, 1113 (2007).
[10] S. Fromang et al., Astron. Astrophys. **476**, 1123 (2007).
[11] G. L. Eyink, Physica D **207**, 91 (2005).
[12] G. Eyink and H. Aluie, Phys. Fluids **21**, 115107 (2009).
[13] H. Aluie and G. Eyink, Phys. Fluids **21**, 115108 (2009).
[14] M. Germano, J. Fluid Mech. **238**, 325 (1992).
[15] C. Meneveau and J. Katz, Annu. Rev. Fluid Mech. **32**, 1 (2000).
[16] H. Aluie and G. Eyink (in preparation).
[17] W.-C. Müller, D. Biskamp, and R. Grappin, Phys. Rev. E **67**, 066302 (2003).
[18] P. D. Mininni and A. Pouquet, Phys. Rev. E **80**, 025401 (2009).
[19] P. Hily-Blant, E. Falgarone, and J. Pety, Astron. Astrophys. **481**, 367 (2008).
[20] C. Salem et al., Astrophys. J. **702**, 537 (2009).
[21] H. Politano and A. Pouquet, Geophys. Res. Lett. **25**, 273 (1998).
[22] A. A. Schekochihin, S. A. Boldyrev, and R. M. Kulsrud, Astrophys. J. **567**, 828 (2002).